

PUSHFORWARDS OF TILTING SHEAVES

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ABSTRACT. We investigate the behaviour of tilting sheaves under pushforward by a finite Galois morphism. We determine conditions under which such a pushforward of a tilting sheaf is a tilting sheaf. We then produce some examples of Severi Brauer flag varieties and arithmetic toric varieties in which our method produces a tilting sheaf, adding to the list of positive results in the literature. We also produce some counterexamples to show that such a pushforward need not be a tilting sheaf.

1. INTRODUCTION

The purpose of this paper is to study push forwards of tilting sheaves. We consider the following setup : given a variety Y with tilting sheaf \mathcal{T} defined over some prime subfield and another variety X defined over k that is an l/k -form of Y . Here l/k is a Galois extension of fields. There is a projection

$$\pi : Y_l \rightarrow X.$$

We investigate when the push forward $\pi_*(\mathcal{T}_l)$ is a tilting sheaf on X . In previous work a number of positive results were obtained, see [Y] and [Na] for certain homogeneous varieties and towers of homogeneous varieties. In this work we give a counterexample to show that these sheaves need not be tilting sheaves in general, see section 5.

As mentioned in the previous paragraph, various positive results regarding tilting sheaves on twisted forms of varieties have been obtained in recent papers. In [Bl], tilting sheaves are constructed on generalized Severi-Brauer varieties via a different approach to that given in this paper. The thesis [Y], constructs tilting bundles on Severi-Brauer schemes and some arithmetic toric varieties using the procedure in this work. More recently, these ideas have been extended to generalized Severi-Brauer schemes and positive characteristic in [Na].

A more detailed overview of the paper follows. In section 2 we recall some basic facts about global dimension of rings. We investigate the behaviour of global dimension under base change. A notion of geometric finite global dimension is introduced. This notion is motivated by the following question: If R is a k -algebra of finite global dimension and l a finite field extension of k , then does the algebra R_l have finite global dimension? See (2.7) and the discussion after it. In section 3, we discuss basic results about tilting sheaves, generation in derived categories and exceptional sequences. A criterion for $\pi_*(\mathcal{T}_l)$ to be a tilting sheaf on Y is given, see (3.4). In section 4 we recall Kapranov's exceptional collection on flag varieties from ([K2]). The section ends by noting that Kapranov's exceptional collection produces a tilting sheaf on any inner form of a partial flag variety, these are also known as Severi Brauer flag varieties. This generalises a result of [Bl], see also [Y] and [Na]. In section 5, we show that if we consider outer forms of flag varieties then

the pushforward does not produce a tilting sheaf. The final section shows how a tilting sheaf can be constructed on certain kinds of arithmetic toric varieties.

NOTATIONS AND CONVENTIONS

We will work over a ground field k of characteristic 0. We need the characteristic 0 assumption in order to make use of the theorem of Borel-Bott-Weil. We will have occasion to make use of possibly non-commutative k -algebras. This notion means that k is in the center of the algebra. We will assume all rings have identities and all modules over them are unital.

2. FINITE GLOBAL DIMENSION

2.1. Background. Let A be a left R -module. Recall that the *projective dimension* of A is the smallest integer n such that there exists a projective resolution

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0.$$

If no such integer exists we define the projective dimension to be ∞ . The projective dimension is denoted by $\text{pd}_R(A)$.

Proposition 2.1. *For a left R -module the following are equivalent :*

- (i) $\text{pd}_R(A) \leq d$.
- (ii) $\text{Ext}^n(A, B) = 0$ for all left R -modules B and $n > d$.

Proof. It is clear that (i) \implies (ii). To see the converse consider a resolution

$$0 \rightarrow M_d \rightarrow P_{d-1} \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0.$$

with P_i projective. Observe that $\text{Ext}^n(M_d, B) = 0$ for all B and $n > 0$. In other words, $\text{Hom}_R(M_d, -)$ is an exact functor. This implies that M_d is projective. \square

Theorem 2.2. *For any ring R the following numbers are equal*

- (1) $\sup\{\text{pd}_R(A) | A \in R\text{-mod}\}$
- (2) $\sup\{\text{pd}_R(R/I) | I \text{ a left ideal}\}$

Proof. See [W, Theorem 4.1.2] \square

The common number is called the *global dimension of R* . If it is finite, we say that R is of *finite global dimension*. We denote this number by $\text{gl. dim } R$.

2.2. Base change for Ext. Throughout this section R will be a k -algebra with unit. Note that this implies that k is in the center of R . Given a field extension l/k we denote by R_l the l -algebra $R \otimes_k l$. For a left R -module A , we denote by A_l the R_l module $l \otimes_k A$. As l is contained in the center of R_l we have that $\text{Hom}_R(A, B_l)$ is naturally a l -vector space when B is a left R -module.

Lemma 2.3. *Let A and B be left R -modules and l/k a field extension. Then there is an isomorphism of l -vector spaces*

$$\Phi : \text{Hom}_R(A, B_l) \rightarrow \text{Hom}_{R_l}(A_l, B_l).$$

Proof. Let $f \in \text{Hom}_R(A, B_l)$, $x \in l$ and $a \in A$. One checks by the universal property of tensor product that there is an R_l -linear map $\Phi(f)$ with

$$\Phi(f)(x \otimes a) = xf(a).$$

Then Φ is an l -linear function. As $A \hookrightarrow A_l$ we see that Φ is injective. Finally for $g \in \text{Hom}_{R_l}(A_l, B_l)$ we have $g = \Phi(g|_A)$ where the restriction is via the inclusion $A \hookrightarrow A_l$. \square

Corollary 2.4. *We have a natural isomorphism*

$$\Phi : \text{Ext}_R^i(A, B_l) \rightarrow \text{Ext}_{R_l}^i(A_l, B_l).$$

Proof. We have δ -functors

$$\text{Ext}_R^i((-), B_l) \text{ and } \text{Ext}_{R_l}^i((-)_l, B_l) : R\text{-mod} \rightarrow l\text{-mod}$$

that agree for $i = 0$. The result follows by observing that they are both coeffaceable as both vanish on free modules for $i > 0$. \square

Let A be a finitely generated R -module. Recall that we have a canonical isomorphism

$$\text{Hom}(A, \oplus_{i \in I} A_i) \cong \oplus_{i \in I} \text{Hom}(A, B_i)$$

for finitely generated modules.

Lemma 2.5. *Suppose that l/k is a field extension and A is finitely generated then there is an isomorphism*

$$\Lambda : l \otimes_k \text{Hom}_R(A, B) \rightarrow \text{Hom}_R(A, B_l).$$

Proof. There is a bilinear pairing

$$l \times \text{Hom}_R(A, B) \rightarrow \text{Hom}_R(A, B_l)$$

sending (x, f) to the homomorphism $a \mapsto xf(a)$ that induces Λ . The l -linearity of Λ is clear. Choose a basis $(e_i)_{i \in I}$ for l/k . This identifies the left hand side with

$$l \otimes_k \text{Hom}_R(A, B) \cong \oplus_{i \in I} \text{Hom}_R(A, B)$$

and the right hand side with

$$\text{Hom}_R(A, B_l) \cong \oplus_{i \in I} \text{Hom}_R(A, B).$$

The first module is isomorphic to the second and Λ realises this isomorphism. \square

Proposition 2.6. *Let R be a left Noetherian k -algebra and A, B be left R -modules. If A is finitely generated then for any field extension we have a canonical isomorphism*

$$l \otimes_k \text{Ext}_R^i(A, B) \rightarrow \text{Ext}_{R_l}^i(A_l, B_l).$$

Proof. We may find a resolution of A by finitely generated R modules, that is of the form:

$$\rightarrow R^{n_2} \rightarrow R^{n_1} \rightarrow R^{n_0} \rightarrow A \rightarrow 0,$$

This resolution can be used to compute $\text{Ext}_R^i(A, B_l)$ and $\text{Ext}_{R_l}^i(A_l, B_l)$. The previous lemma combined with (2.4) gives us the required result. \square

2.3. Geometric global dimension.

Proposition 2.7. *Let R be a left noetherian k -algebra and l/k a field extension. If R_l has finite global dimension then so does R .*

Proof. Suppose that R_l has global dimension d . By (2.2) it suffices to show that R/I has projective dimension smaller than d for each left ideal I . Note that R/I is finitely generated so we may apply (2.6) to see that

$$l \otimes_k \text{Ext}_R^i(R/I, B) = 0$$

for $i > d$ and each R -module B . Hence $\text{Ext}_R^i(R/I, B) = 0$ in for $i > d$. The result follows from (2.1). \square

We do not know how to prove the converse nor do we have a counterexample.

Definition 2.8. We say that a k -algebra R is *geometrically of finite global dimension* if $R_{\bar{k}}$ has finite global dimension for some algebraic closure \bar{k} of k .

2.4. Formal matrix rings. Let T and U be rings and M a (U, T) -bimodule. That is, M is a left T -module and a right U -module such that

$$(um)t = u(mt) \quad \text{for all } u \in U, m \in M, t \in T.$$

Then there is a natural ring structure on the matrices

$$\left\{ \begin{pmatrix} t & 0 \\ m & u \end{pmatrix} : t \in T, u \in U, m \in M \right\}.$$

We will denote this ring by

$$\Lambda_{M,U}^T = \begin{pmatrix} T & 0 \\ M & U \end{pmatrix}$$

and call it the (U, T, M) - formal matrix ring. We will now describe the category of left modules over this ring.

We denote by $\Omega_{M,U}^T$ the category whose objects are triples (A, B, f) where A is a left T -module B is a left U -module and f is a T -morphism

$$M \otimes_R A \rightarrow B.$$

A morphism from (A, B, f) to (A', B', f') is a pair of morphisms (α, β) where $\alpha : A \rightarrow A'$ and $\beta : B \rightarrow B'$ are module maps such that the following diagram commutes

Given a triple (A, B, f) we can define a $\Lambda_{M,U}^T$ -module with underlying abelian group $A \oplus B$ and $\Lambda_{M,U}^T$ -action given by

$$\begin{pmatrix} t & 0 \\ m & u \end{pmatrix} (a, b) = (ta, f(m \otimes a) + ub).$$

This extends to a functor $F_{M,U}^T : \Omega_{M,U}^T \rightarrow \Lambda_{M,U}^T\text{-mod}$.

Proposition 2.9. *The above functor is an equivalence of categories.*

Proof. See [FGR]. \square

Theorem 2.10. *The module $F_{M,U}^T(A, B, f)$ is projective if and only if A is a projective T -module, f is monic and B decomposes as $B = f(M \otimes_T A) \oplus P$ with P a projective U -module.*

Proof. See [HV, Theorem 3.1]. \square

Proposition 2.11. *Suppose that $\text{gl. dim}(T) = m < \infty$ and $\text{gl. dim}(U) = n < \infty$. Then $\text{gl. dim}(\Lambda_{M,U}^T) \leq m + n + 1$.*

Proof. Write $\Lambda = \Lambda_{M,U}^T$. We need to show that for any triple (A, B, f) we have $\text{pd}_\Lambda(A, B, f) \leq m + n + 1$. We have a short exact sequence

$$0 \rightarrow (0, \ker(f), 0) \rightarrow (A, M \otimes_T A, 1_{M \otimes_T A}) \rightarrow (A, \text{im}(f), f) \rightarrow 0.$$

Using (2.10), we have $\text{pd}_\Lambda(0, \ker(f), 0) \leq n$ and $\text{pd}_\Lambda(A, M \otimes_T A, 1_{M \otimes_T A}) \leq m + n + 1$. It follows that

$$\text{pd}_\Lambda(A, \text{im}(f), f) \leq m + n + 1.$$

We have another short exact sequence

$$0 \rightarrow (A, \text{im}(f), f) \rightarrow (A, B, f) \rightarrow (0, B/\text{im}(f), 0) \rightarrow 0.$$

Applying similar arguments to the first part yields the result. \square

3. TILTING SHEAVES AND BASE CHANGE

3.1. Generation in derived categories. Let \mathbf{D} be a triangulated category and S a set of objects in \mathbf{D} . We denote by $\langle S \rangle$ the smallest full triangulated category containing all the objects in S . We denote by $\langle S \rangle^\kappa$ the smallest thick triangulated category containing all the objects in S . Note that thick subcategories are assumed to be full.

An object C of \mathbf{D} is said to be *compact* if $\text{Hom}(C, -)$ commutes with direct sums. We denote by \mathbf{D}^c the full subcategory of compact objects.

Given a set S of objects of \mathbf{D} we define S^\perp to be the full subcategory of \mathbf{D} consisting of objects A with $\text{Hom}_{\mathbf{D}}(E[i], A) = 0$ for all $E \in S$ and $i \in \mathbb{Z}$. We say that S is a *right spanning class* if $S^\perp = 0$.

If \mathbf{D}^c right spans \mathbf{D} we say that \mathbf{D} is compactly generated.

Theorem 3.1. (*Ravenel and Neeman*) *Let \mathbf{D} be a compactly generated triangulated category. Then a set of compact objects S right spans \mathbf{D} if and only if $\langle S \rangle^\kappa = \mathbf{D}^c$.*

Proof. See [BV, Theorem 2.1.2]. \square

Let X be a scheme. We denote by $\text{D}(\mathbf{Qcoh}(X))$ the unbounded derived category of quasi-coherent sheaves on X and by $\text{D}^b(X)$ the bounded derived category of coherent sheaves.

Proposition 3.2. *Let X be a quasi-compact, separated scheme. Then $\text{D}(\mathbf{Qcoh}(X))$ is compactly generated.*

Proof. See [Ne, proposition 2.5]. \square

A complex in $\text{D}(\mathbf{Qcoh}(X))$ is said to be *perfect* if it is locally quasi-isomorphic to a bounded complex of free sheaves.

Proposition 3.3. *Let X be a smooth projective variety. The $C \in \text{D}(\mathbf{Qcoh}(X))$ is compact if and only if C is perfect.*

Proof. See [C, Lemma 3.5]. \square

Proposition 3.4. *Let X/k be a smooth projective variety and l/k a finite field extension. We have canonical morphism $\pi_l : X_l \rightarrow X$. Suppose that \mathcal{T} is a locally free sheaf on X . Then $\langle \mathcal{T} \rangle^\kappa = \mathbf{D}^b(X)$ if and only if $\langle \pi_l^* \mathcal{T} \rangle^\kappa = \mathbf{D}^b(X_l)$.*

Proof. First suppose that $\langle \mathcal{T} \rangle^\kappa = \mathbf{D}^b(X)$. As the functor π_{l*} is exact we have that for each coherent sheaf \mathcal{F} on X_l that $\pi_l^* \pi_{l*} \mathcal{F} \in \pi_l^* \langle \mathcal{T} \rangle^\kappa$. But then by exactness of π_l^* we have that $\mathcal{F} \otimes_k l \in \langle \pi^* \mathcal{T} \rangle^\kappa$. The result follows as \mathcal{F} is a direct summand of $\mathcal{F} \otimes_k l$.

Conversely assume that $\langle \pi^* \mathcal{T} \rangle^\kappa = \mathbf{D}^b(X_l)$. By (3.1), (3.2) and (3.3), it suffices to show that $\mathcal{T}^\perp = 0$. Consider the cartesian square

$$\begin{array}{ccc} X_l & \xrightarrow{\pi_l} & X \\ \downarrow q & & \downarrow p \\ \mathrm{Spec}(l) & \xrightarrow{u} & \mathrm{Spec}(k) \end{array}$$

Suppose that $\mathcal{M} \in \mathcal{T}^\perp$. Then

$$\begin{aligned} 0 &= u^* \mathbf{R} \mathrm{Hom}(\mathcal{T}, \mathcal{M}) \\ &= u^* \mathbf{R} p_*(\mathcal{T}^\vee \otimes \mathcal{M}) \\ &= \mathbf{R} q_* \pi_l^*(\mathcal{T}^\vee \otimes \mathcal{M}) \\ &= \mathbf{R} \mathrm{Hom}(\pi_l^* \mathcal{T}, \pi_l^* \mathcal{M}). \end{aligned}$$

Hence $\pi_l^* \mathcal{M} = 0$ by (3.1), (3.2) and (3.3). Finally $\mathcal{M} = 0$ as π_l is faithfully flat. \square

3.2. Self extensions. Recall that a coherent sheaf \mathcal{F} is said to have no *higher self extensions* if $\mathrm{Ext}^i(\mathcal{F}, \mathcal{F}) = 0$ for $i > 0$.

Lemma 3.5. *Let X/k be a smooth projective variety and l/k a finite field extension. We have canonical morphism $\pi_l : X_l \rightarrow X$. If \mathcal{T} is a locally free coherent sheaf on X then \mathcal{T} has no higher self extensions if and only if $\pi_l^* \mathcal{T}$ has no higher self extensions.*

Proof. This follows via flat base change. \square

3.3. Tilting sheaves and base change.

Definition 3.6. A coherent sheaf \mathcal{F} on X is said to be a *tilting sheaf* if

- (i) $\mathrm{End}(\mathcal{F})$ has finite global dimension
- (ii) \mathcal{F} has no higher self extensions
- (iii) \mathcal{F} generates $\mathbf{D}^b(X)$

Proposition 3.7. *Let X/k be a smooth projective variety and l/k a finite field extension. Denote by $\pi_l : X_l \rightarrow X$ the projection. Suppose that \mathcal{T} is a locally free sheaf on X with endomorphism algebra having geometrically finite global dimension. If $\pi_l^* \mathcal{T}$ is a tilting sheaf then so is \mathcal{T} .*

Proof. Firstly note that $\mathrm{End}(\mathcal{T})$ is a finite dimensional k -algebra as X is projective. Hence it is noetherian. Combine (3.5), (3.4) and (2.7). \square

3.4. Galois Descent. Consider a variety Y defined over the prime subfield k^{pr} of k . Further consider a finite Galois extension l/k with Galois group $\text{Gal}(l/k)$. Let X be an l/k form of Y . This means that X is a variety defined over k and we have an l -isomorphism $X_l \cong Y_l$. Both of the varieties Y_l and X_l have actions of $\text{Gal}(l/k)$. Taking the “difference” of these two actions produces a Galois cocycle

$$\phi_X : \text{Gal}(l/k) \rightarrow \text{Aut}_l(Y_l).$$

The sheaf

$$\mathcal{F} = \bigoplus_{g \in \text{Gal}(l/k)} \phi_X(g)^* \mathcal{T}$$

descends to a sheaf on X . In fact it is just $\pi^* \pi_*(\mathcal{T}_l)$ of \mathcal{T}_l where

$$\pi : Y_l \rightarrow X$$

is the projection. It is clear that \mathcal{F} generates $D^b(X)$, as a direct summand of it is a generator. Hence to see if $\pi_*(\mathcal{T}_l)$ is a tilting sheaf on X , we just need to check the following two properties :

- (T1) the sheaf \mathcal{F} has finite global dimension
- (T2) the sheaf \mathcal{F} has no higher self extensions

In some cases the following result applies :

Proposition 3.8. *In the above setting, suppose that there is a locally free tilting sheaf \mathcal{T} on Y that is geometrically of finite global dimension. Suppose that for each $g \in \text{Gal}(l/k)$ we have*

$$\phi_X(g)^*(\mathcal{T}) \cong \mathcal{T}$$

then there is a tilting sheaf on X obtained by pushing the tilting sheaf on Y_l forward along the projection

$$Y_l \rightarrow X$$

Proof. The condition (T2) is immediate by base change, i.e (3.5). The condition (T1) follows from (2.11). \square

3.5. Tilting sheaves and exceptional collections. Many of the tilting sheaves in this work come from exceptional collections. We begin by recalling the definition.

Definition 3.9. Let \mathbf{D} be a k -linear triangulated category. An object E is said to be *exceptional* if

$$\text{Hom}(E, E) = k \text{ and } \text{Hom}(E, E[m]) = 0 \ \forall \ m \neq 0.$$

An *exceptional collection* in \mathbf{D} is an ordered collection (E_0, E_1, \dots, E_n) of exceptional objects, satisfying

$$\text{Hom}(E_j, E_i[m]) = 0 \text{ for all } m \text{ when } 0 \leq i < j \leq n.$$

If in addition

$$\text{Hom}(E_j, E_i[m]) = 0 \text{ for } 0 \leq j \leq i \leq n, \ m \neq 0,$$

we call (E_0, E_1, \dots, E_n) a *strong exceptional collection*. The collection is *full* (or *complete*) if it generates \mathbf{D} .

Lemma 3.10. *Let $(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n)$ be a full strong exceptional collection of coherent sheaves on X , then $\mathcal{T} = \bigoplus_{i=0}^n \mathcal{F}_i^{\oplus l_i}$, $l_i \geq 1$, is a tilting sheaf on X .*

Proof. It is clear that the constructed sheaf generates the derived category and has no higher self extensions. The statement about global dimension follows from (2.11). \square

4. PARTIAL FLAG VARIETIES

For a fixed k -vector space we will denote by $F(d_1, \dots, d_s, V)$ the partial flag variety of flags

$$V_1 \subseteq V_2 \subseteq \dots \subseteq V_s \subseteq V$$

with $\dim V_i = d_i$. The universal tautological flag will be denoted by

$$\mathcal{W}_{d_1}^{\text{univ}} \subseteq \mathcal{W}_{d_2}^{\text{univ}} \subseteq \dots \subseteq \mathcal{W}_{d_s}^{\text{univ}}.$$

4.1. Kapranov's exceptional collection for partial flag varieties. In [K2] a complete exceptional collection for the partial flag variety $F = F(d_1, \dots, d_s, V)$ is constructed. In this subsection we will describe this collection.

Each such partial flag variety can be expressed as the composite of relative Grassmann bundles. Let $p_r : F(d_r, \dots, d_s, V) \rightarrow F(d_{r+1}, \dots, d_s, V)$ be the natural fibration with fibre $\text{Gr}(d_r, \mathcal{W}_{d_{r+1}}^{\text{univ}})$ for $r = 1, \dots, s$ which we will identify with the relative Grassmann bundle

$$p_r : \text{Gr}(d_r, \mathcal{W}_{d_{r+1}}^{\text{univ}}) \rightarrow F(d_{r+1}, \dots, d_s, V)$$

For each $r = 1, \dots, s$, let Γ_r be the set of all partitions corresponding to Young diagrams fitting into a box of size $d_r \times (d_{r+1} - d_r)$. Then Kapranov's exceptional collection for the partial flag variety $F(d_1, \dots, d_s, V)$ is given by

$$\{\Sigma^{\alpha_1}(\mathcal{W}_{d_1}^{\text{univ}}) \otimes \dots \otimes \Sigma^{\alpha_s}(\mathcal{W}_{d_s}^{\text{univ}}) : \alpha_r \in \Gamma_r, 1 \leq r \leq s\}$$

Note that this exceptional collection is built from the exceptional collection on $\text{Gr}(d_s, V)$ using the sequence of relative Grassmann bundles used to determine the partial flag variety on V .

Theorem 4.1. *The sheaves $\Sigma^{\alpha_1} \mathcal{W}_{d_1}^{\text{univ}} \otimes \dots \otimes \Sigma^{\alpha_s} \mathcal{W}_{d_s}^{\text{univ}}$ occurring in the above decomposition form a complete, strong, exceptional collection for the partial flag variety $F(d_1, \dots, d_s, V)$.*

Proof. See [K2]. □

4.2. Twisted Automorphisms of General Flag Varieties. Let V be an n dimensional k -vector space. Given $1 \leq d_1 \leq \dots \leq d_s \leq n$, we denote by $F = F(d_1, \dots, d_s, V)$ the variety of partial flags of type (d_1, \dots, d_s) in the n dimensional vector space V . When we want to make the base field clear we will write $F(d_1, \dots, d_s, V)_k$ or F_k . Recall that the partial flag variety is a moduli space. As such, there are universal exact sequences

$$0 \rightarrow \mathcal{W}_{d_1}^{\text{univ}} \hookrightarrow \dots \hookrightarrow \mathcal{W}_{d_s}^{\text{univ}} \hookrightarrow \mathcal{O}_F \otimes V \rightarrow \mathcal{Q}_{d_1}^{\text{univ}} \twoheadrightarrow \dots \twoheadrightarrow \mathcal{Q}_{d_s}^{\text{univ}} \rightarrow 0.$$

We begin by recalling the structure of $\text{Aut}_k(F(d_1, \dots, d_s, V))$. Any $\phi \in \text{GL}(V)$ induces new universal exact sequences by

$$0 \rightarrow \mathcal{W}_{d_1}^{\text{univ}} \hookrightarrow \dots \hookrightarrow \mathcal{W}_{d_s}^{\text{univ}} \rightarrow \mathcal{O}_F \otimes V \xrightarrow{1 \otimes \phi} \mathcal{Q}_{d_1}^{\text{univ}} \twoheadrightarrow \dots \twoheadrightarrow \mathcal{Q}_{d_s}^{\text{univ}} \rightarrow 0$$

and hence determines an automorphism of $F(d_1, \dots, d_s, V)$. This gives an inclusion $\text{PGL}(V) \hookrightarrow \text{Aut}_k(F(d_1, \dots, d_s, V))$. In most cases this completely describes the automorphism group. When $d_i + d_{s-i+1} = n$ for all $1 \leq i \leq s$ there is one more automorphism. Choose an isomorphism $V \cong V^\vee$. This induces an automorphism σ of $F(d_1, \dots, d_s, V)$ sending the above universal exact sequences to

$$0 \rightarrow \mathcal{Q}_{d_s}^{\text{univ}^\vee} \hookrightarrow \dots \hookrightarrow \mathcal{Q}_{d_1}^{\text{univ}^\vee} \hookrightarrow \mathcal{O}_F \otimes V \rightarrow \mathcal{W}_{d_1}^{\text{univ}^\vee} \twoheadrightarrow \dots \twoheadrightarrow \mathcal{W}_{d_s}^{\text{univ}^\vee} \rightarrow 0.$$

So in particular, $\sigma^*(\mathcal{W}_{d_i}^{\text{univ}}) \cong \mathcal{Q}_{d_{s-i+1}}^{\text{univ}^\vee}$, for all $i = 1, \dots, s$.

Theorem 4.2. (i) Suppose that there exists i with $d_i + d_{s-i+1} \neq n$. Then $\text{Aut}_k(F(d_1, \dots, d_s, V)) = \text{PGL}(V)$.

(ii) Suppose that for all $1 \leq i \leq s$, we have $d_i + d_{s-i+1} = n$. Then

$$\text{Aut}_k(F(d_1, \dots, d_s, V)) = \langle \text{PGL}(V), \sigma \rangle.$$

Proof. This theorem is due to Chow in characteristic 0, see [CH]. In arbitrary characteristic a proof can be found in [T]. \square

The scheme $F(d_1, \dots, d_s, V)$ can be defined over \mathbb{Z} , along with its universal exact sequences. Hence for each field k and each automorphism α of k lifts canonically to an automorphism, also denoted α , of $F(d_1, \dots, d_s, V)$.

Proposition 4.3. In the above setting we have $\alpha^*(\mathcal{W}^{\text{univ}}) \cong \mathcal{W}^{\text{univ}}$.

Proof. This is because $\mathcal{W}^{\text{univ}}$ descends to a sheaf over $F(d_1, \dots, d_s)_{\mathbb{Z}}$. \square

Corollary 4.4. Let $\phi : F(d_1, \dots, d_s, V) \rightarrow F(d_1, \dots, d_s, V)$ be a twisted automorphism.

- (i) If $\dim V \neq d_i + d_{s-i+1}$ for some $i = 1, \dots, s$, then $\phi^*(\mathcal{W}_{d_i}^{\text{univ}}) \cong \mathcal{W}_{d_i}^{\text{univ}}$ for all $i = 1, \dots, s$.
- (ii) If $\dim V = d_i + d_{s-i+1}$ for all $i = 1, \dots, s$, then either $\phi^*(\mathcal{W}_{d_i}^{\text{univ}}) \cong \mathcal{W}_{d_i}^{\text{univ}}$ or $\phi^*(\mathcal{W}_{d_i}^{\text{univ}}) \cong \mathcal{Q}_{d_{s-i+1}}^{\text{univ} \vee}$.

Proof. After writing $\phi = \psi \circ \alpha$ where ψ is an automorphism of $F(d_1, \dots, d_s, V)$ and α is an automorphism of k the result follows from the above discussion. \square

4.3. Severi Brauer Flag varieties. Consider a Severi-Brauer flag variety $X = \text{SB}(d_1, \dots, d_s, A) \rightarrow \text{Spec}(k)$ where A is a k -central simple algebra. Such an X is an inner form of a partial flag variety. That is, there is a cartesian square of the form

$$\begin{array}{ccc} F(d_1, \dots, d_s, V) & \rightarrow & X \\ \downarrow & & \downarrow \\ \text{Spec}(l) & \longrightarrow & \text{Spec}(k), \end{array}$$

where l/k is a Galois extension and the 1-cocycle

$$\text{Gal}(l/k) \rightarrow \text{Aut}(F(d_1, d_2, \dots, d_s, V))$$

factors through $\text{PGL}(V)$.

Theorem 4.5. $\text{SB}(d_1, \dots, d_s, A)$ has a locally free tilting sheaf.

Proof. We can just apply (3.8) as it is clear that an inner automorphism preserves the sheaves in the exceptional collection. \square

5. OUTER FORMS OF PARTIAL FLAG VARIETIES.

In this section we consider twisted forms of partial flag varieties

$$F = F(d_1, \dots, d_s, V)$$

satisfying $\dim V = d_i + d_{s-i+1}$ such that the associated Galois cocycle

$$\text{Gal}(l/k) \rightarrow \text{Aut}_l(F)$$

does not factor through $\mathrm{PGL}(V)$. The associated form X of F is called an outer form. X can be realised as a Severi Brauer flag variety $\mathrm{SB}(d_1, \dots, d_s, A)$ for A a k central simple algebra equipped with a unitary involution.

In this case our method does not produce a tilting sheaf. This does not mean a tilting sheaf does not exist although, to the best of our knowledge, no such sheaf exists in the literature at this time.

In this setting, the partial flag variety has an extra automorphism σ that sends the tautological flag

$$\mathcal{W}_1^{\mathrm{univ}} \subseteq \dots \subseteq \mathcal{W}_s^{\mathrm{univ}}$$

to

$$\mathcal{Q}_s^{\mathrm{univ}^\vee} \subseteq \dots \subseteq \mathcal{Q}_1^{\mathrm{univ}}$$

Let $\mathcal{E} = \Sigma^{\alpha_1}(\mathcal{W}_1^{\mathrm{univ}}) \otimes \dots \otimes \Sigma^{\alpha_s}(\mathcal{W}_s^{\mathrm{univ}})$ be a bundle in Kapranov's exceptional collection. Then the image under the extra automorphism is $\sigma^*(\mathcal{E}) = \Sigma^{\alpha_1}(\mathcal{Q}_s^{\mathrm{univ}^\vee}) \otimes \dots \otimes \Sigma^{\alpha_s}(\mathcal{Q}_1^{\mathrm{univ}^\vee})$.

We show that when $d_i + d_{s-i+1} = n$ for all i , the image of Kapranov's exceptional collection under the automorphism group of $F(d_1, \dots, d_s, V)$ cannot be an exceptional collection, in particular higher Ext groups do not vanish. In other words, we will produce bundles \mathcal{F} and \mathcal{G} in Kapranov's exceptional collection such that

$$\mathrm{Ext}_F^i(\sigma^*(\mathcal{F}), \mathcal{G}) \neq 0$$

for some $i > 0$.

We first discuss the methods behind our calculations. Let $\mathcal{E} = \sigma^*(\mathcal{F})^\vee \otimes \mathcal{G}$. Then as the exceptional collection consists of vector bundles, we have

$$\mathrm{Ext}_F^*(\sigma^*(\mathcal{F}), \mathcal{G}) = H^*(F, \mathcal{E})$$

Also, we may factor the structure morphism p of $F(d_1, \dots, d_s, V)$ as a sequence of relative Grassmannian bundles

$$p_i : F(d_i, \dots, d_s, V) \rightarrow F(d_{i+1}, \dots, d_s, V), i = 1, \dots, s-1$$

with p_s being the structure morphism for $F(d_s, V) = \mathrm{Gr}(d_s, V)$. Here we identify p_i with the relative Grassmann bundle

$$p_i : \mathrm{Gr}(d_i, \mathcal{W}_{d_{i+1}}^{\mathrm{univ}}) \rightarrow F(d_{i+1}, \dots, d_s, V)$$

Then, since $p = p_s \circ \dots \circ p_1$, we see that, in the derived category, we have

$$Rp_*(\mathcal{E}) = R(p_s)_* \circ \dots \circ R(p_1)_*(\mathcal{E})$$

Let $\mathcal{E}_i = R(p_i)_* \circ R(p_{i-1})_* \circ \dots \circ R(p_1)_*(\mathcal{E})$ for $i = 1, \dots, s$ and $\mathcal{E}_0 = \mathcal{E}$. At each stage i , we wish to reexpress \mathcal{E}_i in terms of bundles of the form

$$R(p_i)_*(\Sigma^\alpha(\mathcal{W}_{d_{i+1}}/\mathcal{W}_{d_i}) \otimes \Sigma^\beta(\mathcal{W}_{d_i})) \otimes \mathcal{E}'_{i+1}$$

where \mathcal{E}'_{i+1} is a bundle defined over $F(d_{i+1}, \dots, d_s, V)$. To do this, we use exact sequences of bundles derived from the natural sequences

$$0 \rightarrow \mathcal{W}_{d_{i+1}}^{\mathrm{univ}}/\mathcal{W}_{d_{i+1}}^{\mathrm{univ}} \rightarrow \mathcal{Q}_{d_i}^{\mathrm{univ}} \rightarrow \mathcal{Q}_{d_{i+1}}^{\mathrm{univ}} \rightarrow 0$$

We will make use of the tools discussed in the next subsection, particularly Proposition 5.2, relative Borel-Weil-Bott and the projection formula to determine \mathcal{E}_i from \mathcal{E}_{i-1} as a bundle of $F(d_{i+1}, \dots, d_s, V)$.

5.1. Cohomological Tools. Fix a Borel subgroup $B \subseteq \mathrm{GL}_n$. The character group of B , $X(B)$ is the character lattice $X(T)$ of the maximal torus T and so is in bijection with \mathbb{Z}^n . Indeed.

$$X(B) = X(T) = \langle \chi_i : i = 1, \dots, n \rangle \cong \mathbb{Z}^n$$

where χ_i is the i th projection. The dominant Weyl chamber C^+ consists of sequences $\chi = (a_1, a_2, \dots, a_n)$ with $a_1 \geq a_2 \geq \dots \geq a_n$. The irreducible representations of $GL(V)$ are given by $\Sigma^\chi(V)$ for each $\chi \in C^+$ where Σ^χ is the corresponding Schur functor. Note that $(\Sigma^\chi(V))^\vee = \Sigma^{-\chi}(V)$ for $\chi \in C^+$ where $-\chi = (-a_n, \dots, -a_1) \in C^+$ if $\chi = (a_1, \dots, a_n) \in C^+$. There is an action of the Weyl group S_n given by permutation of letters. We denote half the sum of the positive roots by $\rho = (n, n-1, \dots, 1)$. There is a modified action of the Weyl group S_n on the weights \mathbb{Z}^n given by

$$\sigma \cdot \lambda = \sigma(\lambda + \rho) - \rho.$$

Let \mathcal{V} be a vector bundle of rank n over a scheme X and $\pi : \mathrm{Flag}(\mathcal{V}) \rightarrow X$ be the relative full flag bundle over X . Note that there is a GL_n -torsor $T(\mathcal{V}) = \mathrm{Isom}(\mathcal{O}_X^n, \mathcal{V})$ over X . The fibre over a point $x \in X$ is the set of frames at $x \in X$, $\mathrm{Isom}(k^n, \mathcal{V}_x)$ on which GL_n acts freely by precomposition. Then $T(\mathcal{V})/B \cong \mathrm{Flag}(\mathcal{V})$. Each character of B , $\chi \in \mathbb{Z}^n$ produces a line bundle

$$\mathcal{O}(\chi) \cong T(\mathcal{V}) \times_{B, \chi} \mathbb{G}_m$$

If $\chi = (\beta_1, \beta_2, \dots, \beta_n)$ then

$$\mathcal{O}(\chi) \cong \mathcal{W}_1^{-\beta_1} \otimes (\mathcal{W}_2/\mathcal{W}_1)^{-\beta_2} \otimes \dots \otimes (\mathcal{V}/\mathcal{W}_{n-1})^{-\beta_n}.$$

The Borel-Bott-Weil Theorem determines $R\pi_*(\mathcal{O}_F(\chi))$ for $\chi \in C_+$.

Theorem 5.1. (*Borel-Bott-Weil*) *Let \mathcal{V} be a vector bundle over a scheme X and $\pi : \mathrm{Flag}(\mathcal{V}) \rightarrow X$ be the relative full flag bundle over X . Let*

$$0 = \mathcal{W}_0 \subseteq \mathcal{W}_1 \subseteq \mathcal{W}_2 \subseteq \dots \subseteq \mathcal{W}_n = \mathcal{V}.$$

be a universal flag on $\mathrm{Flag}(\mathcal{V})$. For $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$, we define a corresponding line bundle on $\mathrm{Flag}(\mathcal{V})$

$$\mathcal{O}_F(\beta) = \mathcal{W}_1^{\otimes -\beta_1} \otimes (\mathcal{W}_2/\mathcal{W}_1)^{\otimes -\beta_2} \dots \otimes (\mathcal{V}/\mathcal{W}_{n-1})^{-\beta_n}$$

Then for $\chi \in \mathbb{Z}^n$:

- (1) *If there exists a non-identity $w \in S_n$ such that $w \cdot \chi = \chi$ (or equivalently if there is a repeat in $\chi + \rho$) then $R^i \pi_*(\mathcal{O}_F(\chi)) = 0$ for all i .*
- (2) *Otherwise, there exists a unique $w \in S_n$ such that $\alpha = w \cdot \chi \in C^+$. In this case, if $i \neq l(w)$, we have $R^i \pi_*(\mathcal{O}(\chi)) = 0$ and $R^{l(w)} \pi_*(\mathcal{O}_F(\chi)) = \Sigma^\alpha(\mathcal{V})^\vee = \Sigma^{-\alpha}(\mathcal{V})$.*

We will be interested in relative Grassmann bundles over a scheme X . Let \mathcal{V} be a bundle over X and let $p : \mathrm{Gr}(k, \mathcal{V}) \rightarrow X$ be the relative Grassmann bundle and $\pi : \mathrm{Flag}(\mathcal{V}) \rightarrow X$. We wish to express the higher derived functors of p for certain bundles over $\mathrm{Gr}(k, \mathcal{V})$ in terms of the higher derived functors of π for certain line bundles over $\mathrm{Flag}(\mathcal{V})$. This proposition follows from the discussion in [K].

Proposition 5.2. *Suppose we have decreasing sequences*

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k) \quad \text{and} \quad \beta = (\beta_1 \geq \beta_2 \geq \dots \geq \beta_{n-k}).$$

Let \mathcal{V} be a bundle on a scheme X , and let $p : \mathrm{Gr}(k, \mathcal{V}) \rightarrow X$ be the relative Grassmann bundle on \mathcal{V} and let $\pi : \mathrm{Flag}(\mathcal{V}) \rightarrow X$ be the full flag variety. Let \mathcal{W} be the tautological subbundle on $\mathrm{Gr}(k, \mathcal{V})$. Then there is a cartesian diagram

$$\begin{array}{ccc} \mathrm{Flag}(\mathcal{V}) & \longrightarrow & \mathrm{Flag}(\mathcal{V}/\mathcal{W}) \\ \downarrow & & \downarrow q_2 \\ \mathrm{Flag}(\mathcal{W}) & \xrightarrow{q_1} & \mathrm{Gr}(k, \mathcal{V}) \end{array}$$

Further $R\pi_*(\mathcal{O}_F(-\alpha_k, \dots, -\alpha_1, -\beta_{n-k}, \dots, -\beta_1)) = Rp_*(\Sigma^\alpha(\mathcal{W}) \otimes \Sigma^\beta(\mathcal{V}/\mathcal{W}))$.

Proof. The statement on the cartesian diagram follows immediately from the description of the flag varieties as moduli spaces.

By Borel-Weil-Bott, we see that $(q_1)_*(\mathcal{L}_1) = (q_1)_*(\mathcal{O}_{F_1}(-\alpha)) = (\Sigma^{-\alpha}(\mathcal{W}))^\vee = \Sigma^\alpha(\mathcal{W})$ and $R^i(q_1)_*(\mathcal{L}_1) = 0$ for $i > 0$ since $-\alpha$ is dominant if α is dominant. Similarly, we see that $(q_2)_*(\mathcal{L}_2) = (q_2)_*(\mathcal{O}_{F_2}(-\beta)) = \Sigma^\beta(\mathcal{V}/\mathcal{W})$. Since

$$0 = \mathcal{W}_0 \subseteq \mathcal{W}_1 \subseteq \mathcal{W}_2 \subseteq \dots \subseteq \mathcal{W}_n = \mathcal{V}.$$

is a universal flag for the relative full flag bundle $F = \mathrm{Flag}(\mathcal{V})$ with projection $q : F = \mathrm{Flag}(\mathcal{V}) \rightarrow G = \mathrm{Gr}(k, \mathcal{V})$, we see that $q = q_1 \times_G q_2$ and $\mathrm{Flag}(\mathcal{V}) = \mathrm{Flag}(\mathcal{W}) \times_G \mathrm{Flag}(\mathcal{V}/\mathcal{W})$. Then $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2 = \mathcal{O}_{\mathrm{Flag}(\mathcal{W})}(-\alpha) \otimes_G \mathcal{O}_{\mathrm{Flag}(\mathcal{V}/\mathcal{W})}(-\beta) = \mathcal{O}_{\mathrm{Flag}(\mathcal{V})}(-\alpha_k, \dots, -\alpha_1, -\beta_{n-k}, \dots, -\beta_1)$. By the Künneth formula, $q_*(\mathcal{L}_1 \otimes \mathcal{L}_2) = (q_1)_*(\mathcal{L}_1) \otimes (q_2)_*(\mathcal{L}_2) = \Sigma^\alpha(\mathcal{W}) \otimes \Sigma^\beta(\mathcal{V}/\mathcal{W})$ and $R^i q_*(\mathcal{L}_1 \otimes \mathcal{L}_2) = 0$ for $i > 0$.

Now $\pi = p \circ q$ where $p : \mathrm{Gr}(k, \mathcal{V}) \rightarrow X$ and $\pi : \mathrm{Flag}(\mathcal{V}) \rightarrow X$. By the Leray spectral sequence, we see that $R\pi_*(\mathcal{L}) = Rp_* \circ Rq_*(\mathcal{L}) = Rp_*(\Sigma^\alpha(\mathcal{W}) \otimes \Sigma^\beta(\mathcal{V}/\mathcal{W}))$ as required. \square

Corollary 5.3. *Let \mathcal{V} be a bundle on X and $p : \mathrm{Gr}(\mathcal{V}, k) \rightarrow X$, the relative Grassmann bundle. Set $G = \mathrm{Gr}(\mathcal{V}, k)$. Then $Rp_*(\mathcal{O}_G) = \mathcal{O}_X$.*

Proof. $\mathcal{O}_G = \Sigma^0(\mathcal{W}) \otimes \Sigma^0(\mathcal{V}/\mathcal{W})$ where \mathcal{W} is the tautological bundle on G and \mathcal{V}/\mathcal{W} is the tautological quotient bundle. The by the proposition and relative Borel-Weil-Bott,

$$Rp_*(\mathcal{O}_G) = R\pi_*(\mathcal{O}_F(0)) = \Sigma^0(\mathcal{V}) = \mathcal{O}_X$$

\square

Recall also the projection formula:

Proposition 5.4. *Let $p : Y \rightarrow X$ be a morphism of schemes. Let \mathcal{E} be a bundle on Y and let \mathcal{F} be a bundle on X . Then $Rp_*(\mathcal{E} \otimes p^*(\mathcal{F})) = Rp_*(\mathcal{E}) \otimes \mathcal{F}$.*

Lastly, we recall a filtration on exterior algebra bundles determined by a short exact sequence which will later prove helpful:

Proposition 5.5. *Let $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ be an exact sequence of locally free sheaves on a scheme X . Then for any r , there is a finite filtration of $\bigwedge^r(\mathcal{F})$,*

$$\bigwedge^r(\mathcal{F}) = F^0 \supseteq F^1 \supseteq F^2 \supseteq \dots \supseteq F^r \supseteq F^{r+1} = 0$$

with quotients

$$F^p/F^{p+1} \cong \bigwedge^p(\mathcal{F}') \otimes \bigwedge^{r-p}(\mathcal{F}'')$$

for each p .

Proof. Exercise II 5.16(c) in Hartshorne. \square

Corollary 5.6. *Let $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ be an exact sequence of locally free sheaves on a scheme X where \mathcal{F}'' has rank 1. Then we obtain an exact sequence*

$$\bigwedge^r(\mathcal{F}') \rightarrow \bigwedge^r(\mathcal{F}) \rightarrow \bigwedge^{r-1}(\mathcal{F}') \otimes \mathcal{F}'' \rightarrow 0$$

Proof. From the proposition, there is a filtration on $\bigwedge^r(\mathcal{F})$ given by

$$\bigwedge^r(\mathcal{F}) = F^0 \supseteq F^1 \supseteq F^2 \supseteq \dots \supseteq F^r \supseteq F^{r+1} = 0$$

with quotients

$$F^p/F^{p+1} \cong \bigwedge^p(\mathcal{F}') \otimes \bigwedge^{r-p}(\mathcal{F}'')$$

for each p . But in our case, F^p/F^{p+1} vanishes for all $p = 0, \dots, r-2$ since $\bigwedge^{r-p}(\mathcal{F}'') = 0$. So we have $\bigwedge^r(\mathcal{F}) = F^0 = \dots = F^{r-1}$ and $F^{r+1} = 0$. This means that the natural exact sequence

$$0 \rightarrow F^r \rightarrow F^{r-1} \rightarrow F^{r-1}/F^r \rightarrow 0$$

gives the exact sequence

$$\bigwedge^r(\mathcal{F}') \rightarrow \bigwedge^r(\mathcal{F}) \rightarrow \bigwedge^{r-1}(\mathcal{F}') \otimes \mathcal{F}'' \rightarrow 0$$

as required. \square

5.2. Non-vanishing Ext groups. In this subsection we consider a flag variety $\text{Flag}(d_1, d_2, \dots, d_s, V)$. Such a flag variety has an extra automorphism σ . We will find sheaves \mathcal{E} and \mathcal{F} in Kapranov's exceptional collection so that

$$\text{Ext}_F^i(\sigma^*(\mathcal{F}), \mathcal{G}) \neq 0$$

for some $i > 0$. It follows that the pushforward of a tilting sheaf to an outer form of a flag variety is not a tilting sheaf.

We will simplify notation a little. The universal subbundle $\mathcal{W}_{d_i}^{\text{univ}}$ and universal quotient bundles $\mathcal{Q}_{d_i}^{\text{univ}}$ by \mathcal{W}_{d_i} and \mathcal{Q}_{d_i} . We will also implicitly identify these bundles $\mathcal{W}_{d_j}, \mathcal{Q}_{d_j}$ over $F(d_j, \dots, d_s, V)$ with their pullbacks to $F(d_i, \dots, d_s, V)$ where $i < j$. Given a partition we will often drop trailing zeroes. For example the partition (2) is really the partition (2, 0, ..., 0). Further repeated entries in a partition we be indicated by superscripts, for example, (1^d) is the partition (1, 1, ..., 1) repeated d -times.

The construction is divided into three cases.

Case 1: $d_1 \geq 2$

Note that as we have assumed that $d_i + d_{s-i+1} = n$ for all $i = 1, \dots, s$, this implies that $d_s = n - d_1$. Take $\mathcal{F} = \Sigma^{(1^{d_1-1})}(\mathcal{W}_{d_1})$ and $\mathcal{G} = \Sigma^{(2)}(\mathcal{W}_{d_s})$. Note that

$$\mathcal{F} = \Sigma^{\alpha_1}(\mathcal{W}_{d_1}) \otimes \dots \otimes \Sigma^{\alpha_s}(\mathcal{W}_{d_s})$$

$$\mathcal{G} = \Sigma^{\beta_1}(\mathcal{W}_{d_1}) \otimes \dots \otimes \Sigma^{\beta_s}(\mathcal{W}_{d_s})$$

where $\alpha_1 = (1^{d_1-1})$ and $\alpha_i = 0$ for all $i \neq 1$ and $\beta_s = (2)$, $\beta_i = 0$ for all $i \neq s$. Since $d_1 - 1 \leq d_1$ and $n - d_s = d_1 \geq 2$, these vector bundles are part of the exceptional collection constructed in (4.1). Then

$$\mathrm{Ext}_F^*(\sigma^*(\mathcal{F}), \mathcal{G}) = H^*(F, \Sigma^{(1^{d_1-1})}(\mathcal{Q}_{d_s}) \otimes \Sigma^{(2)}(\mathcal{W}_{d_s}))$$

We factor the structure morphism of $F = F(d_1, \dots, d_s, V)$ into the projection $q : F(d_1, \dots, d_s, V) \rightarrow \mathrm{Gr}(d_s, V)$ and the structure morphism p for $\mathrm{Gr}(d_s, V)$. Then

$$\begin{aligned} H^*(F, \Sigma^{(1^{d_1-1})}(\mathcal{Q}_{d_s}) \otimes \Sigma^{(2)}(\mathcal{W}_{d_s})) &= Rp_*(Rq_*(\Sigma^{(1^{d_1-1})}(\mathcal{Q}_{d_s}) \otimes \Sigma^{(2)}(\mathcal{W}_{d_s}))) \\ &= Rp_*(\Sigma^{(1^{d_1-1})}(\mathcal{Q}_{d_s}) \otimes \Sigma^{(2)}(\mathcal{W}_{d_s})) \end{aligned}$$

where the last line follows from the projection formula as our bundle is defined over $\mathrm{Gr}(d_s, V)$. Then for the structure morphism π of $\mathrm{Flag}(V)$, we may apply Proposition 5.2 to obtain

$$R\pi_*(\mathcal{O}_{\mathrm{Flag}(V)}(\chi)) = H^*(\mathrm{Flag}(V), \mathcal{O}_{\mathrm{Flag}(V)}(\chi))$$

where $\chi = (0, \dots, 0, -2, 0, -1, \dots, -1)$ has the -2 in the d_s th spot, 0 in the $d_s + 1$ spot and the remaining entries -1 , since π is the structure morphism of $\mathrm{Flag}(V)$. For the simple transposition $w = (d_s, d_s + 1) \in S_n$ of length 1, we see that $w(\chi + \rho)$ is dominant and $\alpha = w \cdot \chi = (0, \dots, 0, -1, \dots, -1)$ where the last $d_1 + 1$ entries are -1 . By Borel-Weil-Bott, we obtain

$$H^1(\mathrm{Flag}(V), \mathcal{O}_{\mathrm{Flag}(V)}(\chi)) = \Sigma^{(1^{d_1+1})}(V)$$

So following the chain of isomorphisms, we find that

$$\mathrm{Ext}_F^1(\sigma^*(\mathcal{F}), \mathcal{G}) = \Sigma^{(1^{d_1+1})}(V) \neq 0$$

so that we have found a bundle of Kapranov's exceptional collection and the image of a bundle of Kapranov's exceptional collection which have non-trivial Ext group.

Case 2: $d_1 = 1, d_2 \geq 3$

Note that $d_{s-1} \geq d_2 \geq 3$ and by the symmetry assumption, we have $d_s = n - 1$ and $d_{s-1} = n - d_2$. Take $\mathcal{F} = \Sigma^{(1^{d_2-1})}(\mathcal{W}_{d_2})$ and $\mathcal{G} = \Sigma^{(2)}(\mathcal{W}_{d_{s-1}})$. Note that

$$\begin{aligned} \mathcal{F} &= \Sigma^{\alpha_1}(\mathcal{W}_{d_1}) \otimes \dots \otimes \Sigma^{\alpha_s}(\mathcal{W}_{d_s}) \\ \mathcal{G} &= \Sigma^{\beta_1}(\mathcal{W}_{d_1}) \otimes \dots \otimes \Sigma^{\beta_s}(\mathcal{W}_{d_s}) \end{aligned}$$

where $\alpha_2 = (1^{d_2-1})$ and $\alpha_i = (0)$ for all $i \neq 2$ and $\beta_{s-1} = (2)$, $\beta_i = (0)$ for all $i \neq s - 1$. Since $d_2 \geq 3$ and $d_s - d_{s-1} = (n - 1) - (n - d_2) = d_2 - 1 \geq 2$, these vector bundles are part of the exceptional collection constructed in (4.1). Then

$$\mathrm{Ext}_F^*(\sigma^*(\mathcal{F}), \mathcal{G}) = H^*(F, \mathcal{E})$$

where $\mathcal{E} = \Sigma^{(1^{d_2-1})}(\mathcal{Q}_{d_{s-1}}) \otimes \Sigma^{(2)}(\mathcal{W}_{d_{s-1}})$. We factor the structure morphism of $F = F(d_1, \dots, d_s, V)$ into $q : F(d_1, \dots, d_s, V) \rightarrow F(d_{s-1}, d_s, V)$ and the structure morphism t for $F(d_{s-1}, d_s, V)$. Then

$$H^*(F, \mathcal{E}) = Rt_*(Rq_*(\mathcal{E})) = Rt_*(\mathcal{E}) = H^*(F(d_{s-1}, d_s, V), \mathcal{E})$$

where we use the projection formula and the fact that our bundle \mathcal{E} is defined over $F(d_{s-1}, d_s, V)$. We now factor the structure morphism t for $F(d_{s-1}, d_s, V)$ into the relative Grassmann bundle $p_{s-1} : F(d_{s-1}, d_s, V) \rightarrow F(d_s, V) = \mathrm{Gr}(d_s, V)$ and the structure morphism p_s for $\mathrm{Gr}(d_s, V)$. So

$$H^*(F(d_{s-1}, d_s, V), \mathcal{E}) = R(p_s)_*(R(p_{s-1})_*(\mathcal{E}))$$

We now analyse $R(p_{s-1})_*(\mathcal{E})$: Since $\mathcal{E} = \Sigma^{(1^r)}(\mathcal{Q}_{d_{s-1}}) \otimes \Sigma^{(2)}(\mathcal{W}_{d_{s-1}})$, where $r = d_2 - 1 = d_s - d_{s-1}$, we need to reexpress the bundle $\Sigma^{(1^r)}(\mathcal{Q}_{d_{s-1}})$ in terms of Schur functors of the bundles $\mathcal{W}_{d_s}/\mathcal{W}_{d_{s-1}}$ and \mathcal{Q}_{d_s} . Note that there is a natural exact sequence of bundles:

$$0 \rightarrow \mathcal{W}_{d_s}/\mathcal{W}_{d_{s-1}} \rightarrow \mathcal{Q}_{d_{s-1}} \rightarrow \mathcal{Q}_{d_s} \rightarrow 0$$

Let $\mathcal{B} = \mathcal{Q}_{d_{s-1}}$, $\mathcal{B}' = \mathcal{W}_{d_s}/\mathcal{W}_{d_{s-1}}$ and $\mathcal{B}'' = \mathcal{Q}_{d_s}$. Then since the Schur functor $\Sigma^{(1^r)}$ is in fact \bigwedge^r and by assumption \mathcal{B}'' has rank 1, we may use Corollary 5.6 to obtain an exact sequence

$$0 \rightarrow \bigwedge^r(\mathcal{B}') \rightarrow \bigwedge^r(\mathcal{B}) \rightarrow \bigwedge^{r-1}(\mathcal{B}') \otimes \mathcal{B}'' \rightarrow 0$$

So we have

$$0 \rightarrow \bigwedge^r(\mathcal{W}_{d_s}/\mathcal{W}_{d_{s-1}}) \rightarrow \bigwedge^r(\mathcal{Q}_{d_{s-1}}) \rightarrow \bigwedge^{r-1}(\mathcal{W}_{d_s}/\mathcal{W}_{d_{s-1}}) \otimes \mathcal{Q}_{d_s} \rightarrow 0$$

Tensoring this with $\Sigma^{(2)}(\mathcal{W}_{d_{s-1}})$, we get

$$0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0$$

where

$$\mathcal{E}' = \Sigma^{(2)}(\mathcal{W}_{d_{s-1}}) \otimes \bigwedge^r(\mathcal{W}_{d_s}/\mathcal{W}_{d_{s-1}}) \text{ and } \mathcal{E}'' = \Sigma^{(2)}(\mathcal{W}_{d_{s-1}}) \otimes \bigwedge^{r-1}(\mathcal{W}_{d_s}/\mathcal{W}_{d_{s-1}}) \otimes \mathcal{Q}_{d_s}.$$

We wish to compute $R(p_{s-1})_*(\mathcal{E})$. We note that

$$0 \rightarrow R(p_{s-1})_*(\mathcal{E}') \rightarrow R(p_{s-1})_*(\mathcal{E}) \rightarrow R(p_{s-1})_*(\mathcal{E}'') \rightarrow 0$$

in the derived category.

$$R(p_{s-1})_*(\mathcal{E}') = R(\pi_{s-1})_*(\mathcal{O}_{\text{Flag}(\mathcal{W}_{d_s})}(\chi))$$

where $\pi_{s-1} : \text{Flag}(\mathcal{W}_{d_s}) \rightarrow F(d_s, V)$ is the relative full flag bundle and $\chi = (0, \dots, -2, -1, \dots, -1) \in \mathbb{Z}^{n-1}$ has a -2 in the d_{s-1} position followed by a string of r -1 s. Adding $\rho = (n-1, \dots, 1)$ to χ , we find that we have $n - d_{s-1} - 2$ in both d_{s-1} and $d_{s-1} + 1$ positions and so $R(p_{s-1})_*(\mathcal{E}') = 0$ by the relative version of Borel-Weil-Bott.

We now calculate

$$\begin{aligned} R(p_{s-1})_*(\mathcal{E}) &= R(p_{s-1})_*(\Sigma^{(2)}(\mathcal{W}_{d_{s-1}}) \otimes \bigwedge^{r-1}(\mathcal{W}_{d_s}/\mathcal{W}_{d_{s-1}}) \otimes \mathcal{Q}_{d_s}) \\ &= R(p_{s-1})_*(\Sigma^{(2)}(\mathcal{W}_{d_{s-1}}) \otimes \bigwedge^{r-1}(\mathcal{W}_{d_s}/\mathcal{W}_{d_{s-1}})) \otimes \mathcal{Q}_{d_s} \end{aligned}$$

where the last line follows by the projection formula as \mathcal{Q}_{d_s} is defined over $F(d_s, V)$. Note that

$$R(p_{s-1})_*(\Sigma^{(2)}(\mathcal{W}_{d_{s-1}}) \otimes \bigwedge^{r-1}(\mathcal{W}_{d_s}/\mathcal{W}_{d_{s-1}})) = R(\pi_{s-1})_*(\mathcal{O}(\chi))$$

where $\pi_{s-1} : \text{Flag}(\mathcal{W}_{d_s}) \rightarrow F(d_s, V)$ is the relative full flag and

$$\chi = (0, \dots, 0, -2, 0, -1, \dots, -1) \in \mathbb{Z}^{n-1}$$

has a -2 in the d_{s-1} spot, a 0 in the $d_{s-1} + 1$ spot and $(r - 1)$ -1's in the remaining positions. A similar calculation using (5.1) shows that the above bundle is $\bigwedge^{r+1}(\mathcal{W}_{d_s})[1]$. Putting this together with the above shows that

$$H^*(F, \mathcal{E}) = R(p_s)_* \left(\bigwedge^{r+1} (\mathcal{W}_{d_s})[1] \otimes \mathcal{Q}_{d_s} \right).$$

For the structure morphism π_s of $\text{Flag}(V)$, we see that

$$R(p_s)_* \left(\bigwedge^{r+1} (\mathcal{W}_{d_s})[1] \otimes \mathcal{Q}_{d_s} \right) = R(\pi_s)_* (\mathcal{O}(\chi'))$$

where $\chi' = (0, \dots, 0, -1, \dots, -1) \in \mathbb{Z}^n$ has a string of $r + 1 = d_2 + 1$ -1's at the end. Since this weight is dominant, an application of (5.1) produces $\bigwedge^{r+1}(V)[1]$. Hence

$$\text{Ext}^1(\sigma^*(\mathcal{F}), \mathcal{G}) = \bigwedge^{r+1}(V) \neq 0$$

and all other Ext groups vanish. Note that $r + 1 = d_2 + 1 \leq n$ by assumption so that $\bigwedge^{r+1}(V) \neq 0$.

Case 3: $d_1 = 1, d_2 = 2$.

Note that $n \geq 3$, and we have $d_s = n - 1$ and $d_{s-1} = n - 2$ by the symmetry assumption. Take $\mathcal{F} = \mathcal{W}_1$ and $\mathcal{G} = \mathcal{W}_{n-2} \otimes \mathcal{W}_{n-1}$. Note that

$$\begin{aligned} \mathcal{F} &= \Sigma^{\alpha_1}(\mathcal{W}_{d_1}) \otimes \dots \Sigma^{\alpha_s}(\mathcal{W}_{d_s}) \\ \mathcal{G} &= \Sigma^{\beta_1}(\mathcal{W}_{d_1}) \otimes \dots \Sigma^{\beta_s}(\mathcal{W}_{d_s}) \end{aligned}$$

where $\alpha_1 = (1)$ and $\alpha_i = (0)$ for all $i \neq 1$ and $\beta_{s-1} = \beta_s = (1)$, $\beta_i = (0)$ for all $i \neq s - 1, s$. Since $d_i \geq 1$ for all i , these bundles are clearly in the exceptional collection constructed in (4.1). Then

$$\text{Ext}_F^*(\sigma^*(\mathcal{F}), \mathcal{G}) = H^*(F, \mathcal{E})$$

where $\mathcal{E} = \mathcal{Q}_{n-1} \otimes \mathcal{W}_{n-1} \otimes \mathcal{W}_n$. We factor the structure morphism of $F = F(d_1, \dots, d_s, V)$ into $q : F(d_1, \dots, d_s, V) \rightarrow F(d_{s-1}, d_s, V)$ and the structure morphism t for $F(d_{s-1}, d_s, V)$. The same calculation as in the second case shows that for $F = F(d_1, \dots, d_s, V)$ we have

$$H^*(F, \mathcal{E}) = H^*(F(d_{s-1}, d_s, V), \mathcal{E})$$

since our bundle \mathcal{E} is defined over $F(d_{s-1}, d_s, V) = F(n - 2, n - 1, V)$. We now factor the structure morphism t for $F(n - 2, n - 1, V)$ into the relative Grassmann bundle $p_{s-1} : F(n - 2, n - 1, V) \rightarrow F(n - 1, V) = \text{Gr}(n - 1, V)$ and the structure morphism p_s for $\text{Gr}(n - 1, V)$. So

$$H^*(F(n - 2, n - 1, V), \mathcal{E}) = R(p_s)_* (R(p_{s-1})_*(\mathcal{E}))$$

We now analyse $R(p_{s-1})_*(\mathcal{E})$: Note that we have an exact sequence

$$0 \rightarrow \mathcal{W}_{n-1}/\mathcal{W}_{n-2} \rightarrow \mathcal{Q}_{n-2} \rightarrow \mathcal{Q}_{n-1} \rightarrow 0$$

Tensoring with $\mathcal{W}_{n-2} \otimes \mathcal{W}_{n-1}$ we get an exact sequence

$$0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0$$

where $\mathcal{E}' = (\mathcal{W}_{n-1}/\mathcal{W}_{n-2}) \otimes \mathcal{W}_{n-2} \otimes \mathcal{W}_{n-1}$ and $\mathcal{E}'' = \mathcal{Q}_{n-1} \otimes \mathcal{W}_{n-2} \otimes \mathcal{W}_{n-1}$. We then have

$$0 \rightarrow R(p_{s-1})_*(\mathcal{E}') \rightarrow R(p_{s-1})_*(\mathcal{E}) \rightarrow R(p_{s-1})_*(\mathcal{E}'') \rightarrow 0$$

We first analyse $R(p_{s-1})_*(\mathcal{E}'')$. By the projection formula, we have

$$R(p_{s-1})_*(\mathcal{E}'') = R(p_{s-1})_*(\mathcal{W}_{n-2}) \otimes \mathcal{W}_{n-1} \otimes \mathcal{Q}_{n-1}$$

But

$$R(p_{s-1})_*(\mathcal{W}_{n-2}) = R(\pi_{s-1})_*(\mathcal{O}(\chi))$$

where $\pi_{s-1} : \text{Flag}(\mathcal{W}_{n-1}) \rightarrow F(n-1, V)$ is the relative full flag and

$$\chi = (0, \dots, 0, -1, 0) \in \mathbb{Z}^{n-1}.$$

The last two entries of $\chi + \rho$ are 1 and so by (5.1), we see that $R(p_{s-1})_*(\mathcal{W}_{n-2}) = 0$ and so $R(p_{s-1})_*(\mathcal{E}'') = 0$. Then

$$R(p_{s-1})_*(\mathcal{E}) = R(p_{s-1})_*(\mathcal{E}') = R(p_{s-1})_*((\mathcal{W}_{n-1}/\mathcal{W}_{n-2}) \otimes \mathcal{W}_{n-2}) \otimes \mathcal{W}_{n-1}$$

where the last equality follows from the projection formula. Then

$$R(p_{s-1})_*(\mathcal{W}_{n-2} \otimes (\mathcal{W}_{n-1}/\mathcal{W}_{n-2})) = R(\pi_{s-1})_*(\mathcal{O}(\chi'))$$

where $\pi_{s-1} : \text{Flag}(\mathcal{W}_{n-1}) \rightarrow F(n-1, V)$ is the relative full flag and

$$\chi = (0, \dots, 0, -1, -1) \in \mathbb{Z}^{n-1}.$$

Since χ is dominant, so is $\chi + \rho$, and so by (5.1), we see that

$$R(p_{s-1})_*(\mathcal{W}_{n-2} \otimes \mathcal{W}_{n-1}/\mathcal{W}_{n-2}) = \bigwedge^2(\mathcal{W}_{n-1}).$$

Then by Littlewood Richardson, we have

$$R(p_{s-1})_*(\mathcal{E}') = \bigwedge^2(\mathcal{W}_{n-1}) \otimes \mathcal{W}_{n-1} = \bigwedge^3(\mathcal{W}_{n-1}) \oplus \Sigma^{(2,1)}(\mathcal{W}_{n-1})$$

[Note that here we have $n \geq 3$ and so $\Sigma^{(2,1)}(\mathcal{W}_{n-1}) \neq 0$ but $\bigwedge^3(\mathcal{W}_{n-1}) \neq 0$ if and only if $n \geq 4$. This will turn out not to matter as this term vanishes in the next step.] So

$$R(p_s)_*(R(p_{s-1})_*(\mathcal{E})) = R(p_s)_*(\bigwedge^3(\mathcal{W}_{n-1})) \oplus R(p_s)_*(\Sigma^{(2,1)}(\mathcal{W}_{n-1}))$$

Let π_s be the structure morphism for $\text{Flag}(V)$. Then we have

$$R(p_s)_*(\bigwedge^3(\mathcal{W}_{n-1})) = R(\pi_s)_*(\chi_1)$$

where $\chi_1 = (0, \dots, 0, -1, -1, -1, 0) \in \mathbb{Z}^n$. Here $\chi_1 + \rho$ has a repeat of 1 in the last two entries and so $R(p_s)_*(\bigwedge^3(\mathcal{W}_{n-1})) = 0$ by relative Borel-Weil-Bott. But we also have

$$R(p_s)_*(\Sigma^{(2,1)}(\mathcal{W}_{n-1})) = R(\pi_s)_*(\chi_2)$$

where $\chi_2 = (0, \dots, 0, 0, -1, -2, 0) \in \mathbb{Z}^n$. Here $\chi_2 + \rho = (n, n-1, \dots, 4, 2, 0, 1)$. Letting $w = (n-1, n)$, we see that $\alpha_2 = w \cdot \chi_2 = (0, \dots, 0, -1, -1, -1)$ and so $R(p_s)_*(\Sigma^{(2,1)}(\mathcal{W}_{n-1})) = \bigwedge^3(V)[1]$ by (5.1). Then we have

$$\text{Ext}^1(\sigma_*(\mathcal{F}), \mathcal{G}) = \bigwedge^3(V) \neq 0$$

and all other Ext groups vanish.

6. APPLICATIONS TO ARITHMETIC TORIC VARIETIES

Theorem 6.1. ([Be]) *The derived category $D^b(\mathbb{P}^n)$ is generated by the strong exceptional collection*

$$\{\mathcal{O}(-n), \mathcal{O}(-n+1), \dots, \mathcal{O}(-1), \mathcal{O}\}.$$

Now let us fix some notation. For projective space \mathbb{P}^n , we always choose $\{\mathcal{O}(1)\}$ as a basis of $\text{Pic}(\mathbb{P}^n) = \mathbb{Z}$; for a projective bundle $p : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^n$, we always choose $\{p^*\mathcal{O}(1), \mathcal{O}_{\mathcal{E}}(1)\}$ as a basis of $\text{Pic}(\mathbb{P}(\mathcal{E})) = \mathbb{Z} \oplus \mathbb{Z}$ and we denote by $\mathcal{O}(i, j)$ the tensor product $p^*\mathcal{O}(i) \otimes \mathcal{O}_{\mathcal{E}}(j)$; and so on.

Proposition 6.2. *Consider projective bundle $p : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^n$. Assume $\mathcal{E} = \mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_{r+1}$ such that \mathcal{L}_i is in $\text{Pic}(\mathbb{P}^n)^+ \simeq \mathbb{Z}_{\geq 0}$ for all $1 \leq i \leq r+1$. Then*

$$(\mathcal{O}(-n, -r), \mathcal{O}(-n+1, -r), \dots, \mathcal{O}(0, -r), \dots, \mathcal{O}(-n, 0), \dots, \mathcal{O}(0, 0))$$

is a full strong exceptional collection of coherent sheaves on $\mathbb{P}(\mathcal{E})$.

Proof. By Theorem (6.1) and [Or, Corollary 2.7], we only need to show this set is a strong set, which follows from an easy computation of projective space cohomology. \square

More generally, we have

Corollary 6.3. *Consider a series of projective bundles $\mathbb{P}(\mathcal{E}_m) \rightarrow \dots \rightarrow \mathbb{P}(\mathcal{E}_i) \rightarrow \mathbb{P}^{r_0}$. Assume \mathcal{E}_i is decomposable of rank $r_i + 1$ and all its summands are in $\text{Pic}(\mathbb{P}(\mathcal{E}_{i-1}))^+ \simeq (\mathbb{Z}_{\geq 0})^{\oplus i}$ for all $1 \leq i \leq m$. Then the set*

$$\{\mathcal{O}(j_0, j_1, \dots, j_m) : -r_i \leq j_i \leq 0, 0 \leq i \leq m\}$$

is a full strong exceptional collection of coherent sheaves on $\mathbb{P}(\mathcal{E}_m)$ by the lexicographical order on (j_0, j_1, \dots, j_m) .

Recall that, see [ELST], an *arithmetic torus* over k of rank n is an algebraic group \mathcal{T} over k such that $\mathcal{T}_l \simeq T_{N,l}$ for some finite Galois extension l/k and lattice N of rank n , and an *arithmetic toric variety* over k is a pair (Y, \mathcal{T}) , where \mathcal{T} is an arithmetic torus over k and Y is a normal variety over k equipped with a faithful action of \mathcal{T} which has a dense orbit. Let $(Y_{\Sigma,l}, T_{N,l})$ be its split toric variety and $G = \text{Gal}(l/k)$, then the G -action on $(Y_{\Sigma,l}, T_{N,l})$ is determined by a conjugacy class of group homomorphisms $\varphi : G \rightarrow \text{Aut}(N)$ such that $\varphi(G) \subseteq \text{Aut}_{\Sigma}$.

Definition 6.4. ([B]) Let Σ be a smooth complete fan, we call a nonempty subset $\mathcal{P} = \{x_1, \dots, x_k\} \subseteq \Sigma(1)$ a *primitive collection* if for each element $x_i \in \mathcal{P}$, the remaining elements $\mathcal{P} \setminus \{x_i\}$ generate a $(k-1)$ -dimensional cone in Σ , while \mathcal{P} itself does not generate any k -dimensional cone in Σ . And we say Σ is a *splitting fan* if any two different primitive collections in $\Sigma(1)$ are disjoint.

Theorem 6.5. *Let (X, \mathcal{T}) be an arithmetic toric variety over k , whose split toric variety corresponding to a splitting fan, then there exists a tilting bundle on X .*

Proof. Let X_l be the corresponding split toric variety with splitting fan Σ in a lattice N , where l/k is a Galois extension with Galois group G . By [B, Theorem 4.3], we have a projectivization $X_l = \mathbb{P}(\mathcal{E}) \rightarrow X'_l$, which corresponds to a primitive collection $\mathcal{P} = \{x_1, x_2, \dots, x_{k+1}\} \subseteq \Sigma(1)$ with primitive relation $x_1 + x_2 + \dots + x_{k+1} = 0$ by [B, Proposition 4.1]. Since $\Sigma(1)$ generates Σ , the action of G on Σ is determined by the action of G on $\Sigma(1)$. As G preserves the primitive relationship and \mathcal{P}

has no intersection with any other primitive collection in $\Sigma(1)$, we must have either $g(\mathcal{P}) = \mathcal{P}$ or $g(\mathcal{P}) \cap \mathcal{P} = \emptyset$ for any $g \in G$. Let the distinguished primitive collections $\mathcal{P}_1, \dots, \mathcal{P}_m$ be the images of \mathcal{P} under the action of G . Again, by [B, Proposition 4.1], these primitive collections determine a series of projective bundles $\mathbb{P}(\mathcal{E}_1) \rightarrow \dots \rightarrow \mathbb{P}(\mathcal{E}_m) \rightarrow Y_l$, where Y_l is also a toric variety with splitting fan by [B, Theorem 4.3].

By [O, page 59], we may construct the fan Σ from the fan Σ_{Y_l} . The Galois G -action on X_l induces an Galois G -action on Y_l . Let Y_l descend to (Y_k, \mathcal{T}') . Then we have a compatible commutative diagram:

$$\begin{array}{ccc} X_l & \longrightarrow & (X, \mathcal{T}) \\ \downarrow & & \downarrow \\ Y_l & \longrightarrow & (Y_k, \mathcal{T}'). \end{array}$$

Actually, for every $\tau \in S_m$, the permutation group of the set $\{1, 2, \dots, m\}$, we have a series of projective bundles $\mathbb{P}(\mathcal{E}_{\tau_1}^\tau) \rightarrow \mathbb{P}(\mathcal{E}_{\tau_2}^\tau) \rightarrow \dots \rightarrow \mathbb{P}(\mathcal{E}_{\tau_m}^\tau) \rightarrow Y_L$. Thus each of these primitive collections $\mathcal{P}_1, \dots, \mathcal{P}_m$ induces a projective bundle $\mathbb{P}(\mathcal{E}_i) \rightarrow Y_L, i = 1, \dots, m$. The G -action on X_l induces commutative diagrams

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}_i) & \xrightarrow{\rho_{g_{i,j}}} & \mathbb{P}(\mathcal{E}_j) \\ \downarrow & & \downarrow \\ Y_l & \xrightarrow{\rho_{g_{i,j}}} & Y_l \end{array}$$

for $1 \leq i, j \leq m$. So we may assume that $\{\rho_g^*(\mathcal{E}) : g \in G\} = \{\mathcal{E}_1, \dots, \mathcal{E}_m\}$.

Denote by $X'_l = \mathbb{P}(\mathcal{E}_1) \times_{Y_l} \dots \times_{Y_l} \mathbb{P}(\mathcal{E}_m)$, we can see that $\Sigma_{X_l} \simeq \Sigma_{X'_l}$, and hence $X_l \simeq \mathbb{P}(\mathcal{E}_1) \times_{Y_l} \dots \times_{Y_l} \mathbb{P}(\mathcal{E}_m)$.

Thus we get the following compatible commutative diagram

$$\begin{array}{ccc} X_l = \mathbb{P}(\mathcal{E}_1) \times_{Y_l} \dots \times_{Y_l} \mathbb{P}(\mathcal{E}_m) & \longrightarrow & (X, \mathcal{T}) \\ \downarrow & & \downarrow \\ Y_l & \longrightarrow & (Y_l, \mathcal{T}'), \end{array}$$

where \mathcal{E}_i is a decomposable bundle for all $1 \leq i \leq m$ by [DS, Lemma 1.1].

Iteratively, we get the following diagram:

$$\begin{array}{ccc}
X_{t,l} = \mathbb{P}(\mathcal{E}_{t,1}) \times_{X_{t-1,l}} \cdots \times_{X_{t-1,l}} \mathbb{P}(\mathcal{E}_{t,m_t}) & \longrightarrow & (X_{t,k}, \mathcal{T}_l) = (X, \mathcal{T}) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
X_{2,l} = \mathbb{P}(\mathcal{E}_{2,1}) \times_{X_{1,l}} \cdots \times_{X_{1,l}} \mathbb{P}(\mathcal{E}_{2,m_2}) & \longrightarrow & (X_{2,k}, \mathcal{T}_2) \\
\downarrow & & \downarrow \\
X_{1,l} = \times_l^{m_1} \mathbb{P}(\mathcal{E}_1) & \longrightarrow & (X_{1,k}, \mathcal{T}_1) \\
\downarrow & & \downarrow \\
X_{0,l} = \text{Spec } l & \longrightarrow & \text{Spec } k
\end{array}$$

where \mathcal{E}_1 is a decomposable vector bundle of rank $r_1 + 1$ over $X_{0,L}$ and \mathcal{E}_{i,j_i} is a decomposable vector bundle of rank $r_i + 1$ over $X_{i-1,l}$ and $\{\rho_g^*(\mathcal{E}_{i,1}) : g \in G\} = \{\mathcal{E}_{i,1}, \dots, \mathcal{E}_{i,m_i}\}$ for $2 \leq i \leq t$ and $1 \leq j_i \leq m_i$.

As we know that $\text{Pic}(X_{i,l}) \simeq \mathbb{Z}^{\oplus(m_1+\dots+m_i)}$, we may assume all the line bundle summands of \mathcal{E}_i are in $(\mathbb{Z}_{\geq 0})^{\oplus(m_1+\dots+m_{i-1})}$ for all $2 \leq i \leq t$.

Without causing confusion, we use the same notation $\mathcal{O}_{\mathbb{P}(\mathcal{E}_{i,j})}(s)$ ($1 \leq i \leq t$, $1 \leq j \leq m_i$) to denote the corresponding component in $\text{Pic}(X_{h,L})$ for all $i \leq h \leq t$. Denote by

$$\begin{aligned}
& \mathcal{O}(j_{1,1}, \dots, j_{1,m_1}, \dots, j_{t,1}, \dots, j_{t,m_t}) \\
&= (\mathcal{O}_{\mathbb{P}(\mathcal{E}_1)}(j_{1,1}), \dots, \mathcal{O}_{\mathbb{P}(\mathcal{E}_1)}(j_{1,m_1}), \dots, \mathcal{O}_{\mathbb{P}(\mathcal{E}_{t,1})}(j_{t,1}), \dots, \mathcal{O}_{\mathbb{P}(\mathcal{E}_{t,m_t})}(j_{t,m_t})),
\end{aligned}$$

where $-r_i \leq j_{i,k_i} \leq 0$ and $1 \leq k_i \leq m_i$ for $1 \leq i \leq t$.

Then by (6.3), the set

$$\{\mathcal{O}(j_{1,1}, \dots, j_{1,m_1}, \dots, j_{t,1}, \dots, j_{t,m_t}) : -r_i \leq j_{i,k_i} \leq 0, 1 \leq k_i \leq m_i\}$$

is a full strong exceptional collection of $D^b(X_l)$ by the lexicographical order on $(j_{1,1}, \dots, j_{1,m_1}, \dots, j_{t,1}, \dots, j_{t,m_t})$. For any $g \in G$, we have

$$\begin{aligned}
& \rho_g^* \mathcal{O}(j_{1,1}, \dots, j_{1,m_1}, \dots, j_{t,1}, \dots, j_{t,m_t}) \\
&= \mathcal{O}(j_{1,\tau_{1,g}(1)}, \dots, j_{1,\tau_{1,g}(m_1)}, \dots, j_{t,\tau_{t,g}(1)}, \dots, j_{t,\tau_{t,g}(m_t)}),
\end{aligned}$$

where $\tau_{i,g}$, $1 \leq i \leq t$, are permutations of the corresponding sets $\{1, \dots, m_i\}$. So it is also in the same set.

Let

$$\mathcal{T} = \oplus \rho_g^* \mathcal{O}(j_{1,1}, \dots, j_{1,m_1}, \dots, j_{t,1}, \dots, j_{t,m_t}),$$

then \mathcal{T} is a tilting sheaf on $X_{t,l}$ by Lemma (3.10), and \mathcal{T} descends to a tilting bundle on X by (3.7). \square

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